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The standard solution for surgery

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We define a *standard initial metric* to be a complete and rotationally symmetric metric g_0 on $\mathbb{R}^3$ with nonnegative sectional curvature at every point, constant sectional curvature $\frac{1}{4}$ near the origin $p = 0$ and satisfying the condition that there is a compact ball $B = B_{g_0}(p, A_0)$ such that $g_0|_{\mathbb{R}^3 \setminus B}$ is isometric to the half-cylinder $(S^2, h) \times (\mathbb{R}_+, ds^2)$, where h is the round metric on S^2 with scalar curvature 1.
A *partial standard Ricci flow* is a solution $(\mathbb{R}^3, g(t))_{0 \leq t < T}$ of the Ricci flow starting from a standard initial metric $g(0) = g_0$ and satisfying the property that its curvature is locally bounded in time. Such a partial standard Ricci flow is a *standard Ricci flow* if T is maximal in the sense that there is no extension of the flow past time T which still has curvature locally bounded in time.
In a first step, we prove (via an explicit construction) that there exists a standard initial metric. We then fix once and for all this (or any other) standard initial metric g_0 and prove the following theorem for the corresponding standard Ricci flow.

Theorem 0.1
There exists a standard Ricci flow for some positive amount of time. Let $(\mathbb{R}^3, g(t))_{0 \leq t < T}$ be a standard Ricci flow. Then

- Uniqueness*: If $(\mathbb{R}^3, g'(t))_{0 \leq t < T'}$ is a standard Ricci flow, then $T' = T$ and $g'(t) = g(t)$ for all $t \in [0, T)$.*
- Time interval*: $T = 1$.*
- Rotational symmetry*: For all $t \in [0, 1)$, the metric $g(t)$ is invariant under the $SO(3)$-action on $\mathbb{R}^3$.*
- Positive curvature*: For all $t \in (0, 1)$, the metric $g(t)$ is compete of strictly positive curvature.*
- Asymptotics at infinity*: For all $t_0 < 1$ and $\varepsilon > 0$, there is a compact subset $X \subset \mathbb{R}^3$ such that all x in the complement of X have a neighborhood $U \ni x$ with $g(t)|_U$ being ε-close in $C^{[1/\varepsilon]}$ to $(S^2 \times (-\varepsilon^{-1}, \varepsilon^{-1}), h(t) \oplus ds^2)$, $\forall t \in [0, t_0]$, where $h(t)$ is the round metric on S^2 with scalar curvature $\frac{1}{1-t}$.*
- Curvature bound*: There is $C > 0$ such that for all (x, t) in the standard solution there holds $R(x, t) \geq \frac{C}{1-t}$.*
- Non-collapsing*: There is $r > 0$ and $\kappa > 0$ such that $(\mathbb{R}^3, g(t))$ is κ-non-collapsed on scales less than r for all $t \in [0, 1)$.*
- Canonical neighborhoods*: For all $\varepsilon > 0$ there exists a constant $C = C(\varepsilon)$ such that for all (x, t) in the standard solution one of the following holds*

1. *(x,t) is contained in the core of a (C,ε)-cap.*
2. *(x,t) is the center of an evolving ε-neck with initial time-slice $t=0$ and this time-slice is disjoint from the surgery cap $B_{g_0}(p,A_0+4)$.*
3. *(x,t) is the center of an evolving ε-neck defined for rescaled backwards time at least $1+\varepsilon$.*

Here, we give some remarks about the proof. A complete proof can be obtained by combining Theorem 12.5, Proposition 12.31 and Theorem 12.32 of [3].
Gluing together two copies of $B_{g_0}(p,R)$, $R \geq A_0+1$, at their boundaries yields smooth manifolds (S_R, g_R, p) which converge geometrically to $(\mathbb{R}^3, g_0, p)$ as $R \to \infty$. Since all these manifolds are compact and have the same curvature bounds, existence of a standard solution follows as a geometric limit of the Ricci flows starting from (S_R, g_R, p). Here we used the compactness for Ricci flows from [2]. This argument also implies completeness of the standard solution. Similarly, letting $y_k \to \infty$, we obtain $(S^2,h)\times(\mathbb{R},ds^2)$ as the geometric limit of $(\mathbb{R}^3, g_0, y_k)$ and the claimed asymptotics at infinity follow. Now, positive curvature follows easily by localizing the standard result in dimension three.
One can obtain uniqueness form [1]. This immediately implies rotational symmetry, too. On the other hand, rotational symmetry also follows from the fact that Killing fields are stationary under the flow and stay Killing fields. With this symmetry, one can then reduce the equation to a one-dimensional problem for which uniqueness is obtained from a slightly modified version of DeTurck's trick for compact manifolds. This gives a much simpler proof than the general one in [1].
Non-collapsing of the standard solution is a direct application of the general non-collapsing result, cf. [4], section 8. All the remaining results ($T=1$, curvature bounds, and the canonical neighborhoods statement) can then be proven by contradiction via a blow-up argument. Indeed, if one of the statements fails to hold, one finds a sequence of points for which the blow-up limit is a κ-solution. Using the fact that κ-solutions have canonical neighborhoods and asymptotically vanishing volume, we then obtain the desired contradictions.

References

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